

# Solution of a sum form equation in the two dimensional closed domain case\*

**Imre Kocsis**

Faculty of Engineering University of Debrecen  
e-mail:kocsisi@mfk.unideb.hu

## Abstract

In this note we give the solution of the sum form functional equation

$$\sum_{i=1}^n \sum_{j=1}^m f(p_i \bullet q_j) = \sum_{i=1}^n f(p_i) \sum_{j=1}^m f(q_j)$$

arising in information theory (in characterization of so-called entropy of degree  $\alpha$ ), where  $f : [0, 1]^2 \rightarrow \mathbb{R}$  is an unknown function and the equation holds for all two dimensional complete probability distributions.

**Key Words:** Sum form equation, additive function, multiplicative function.

**AMS Classification Number:** 39B22

## 1. Introduction

In the following we denote the set of real numbers and the set of positive integers by  $\mathbb{R}$  and  $\mathbb{N}$ , respectively. Throughout the paper we shall use the following

notations:  $\underline{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^k$ ,  $\underline{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^k$ . For all  $3 \leq n \in \mathbb{N}$  and for all  $k \in \mathbb{N}$  we define the sets  $\Gamma_n^c[k]$  and  $\Gamma_n^0[k]$  by

$$\Gamma_n^c[k] = \left\{ (p_1, \dots, p_n) : p_i \in [0, 1]^k, i = 1, \dots, n, \sum_{i=1}^n p_i = \underline{1} \right\}$$

---

\*This research has been supported by the Hungarian National Foundation for Scientific Research (OTKA), grant No. T-030082.

and

$$\Gamma_n^0[k] = \left\{ (p_1, \dots, p_n) : p_i \in ]0, 1[^k, i = 1, \dots, n, \sum_{i=1}^n p_i = \underline{1} \right\},$$

respectively.

$$\text{If } x = \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix}, y = \begin{pmatrix} y_1 \\ \vdots \\ y_k \end{pmatrix} \in \mathbb{R}^k \text{ then } x \bullet y = \begin{pmatrix} x_1 y_1 \\ \vdots \\ x_k y_k \end{pmatrix} \in \mathbb{R}^k.$$

If we do not say else we denote the components of an element  $P$  of  $\Gamma_n^c[2]$  or  $\Gamma_n^0[2]$  by

$$P = (p_1, \dots, p_n) = \begin{pmatrix} p_{11} & \dots & p_{n1} \\ p_{12} & \dots & p_{n2} \end{pmatrix}.$$

A function  $A : \mathbb{R}^k \rightarrow \mathbb{R}$  is additive if  $A(x + y) = A(x) + A(y)$ ,  $x, y \in \mathbb{R}^k$ , a function  $M : ]0, 1[^k \rightarrow \mathbb{R}$  is multiplicative if  $M(x \bullet y) = M(x)M(y)$ ,  $x, y \in ]0, 1[^k$ , a function  $M : [0, 1]^k \rightarrow \mathbb{R}$  is multiplicative if  $M(\underline{0}) = 0$ ,  $M(\underline{1}) = 1$ , and  $M(x \bullet y) = M(x)M(y)$ ,  $x, y \in [0, 1]^k$ .

The functional equation

$$\sum_{i=1}^n \sum_{j=1}^m f(p_i \bullet q_j) = \sum_{i=1}^n f(p_i) \sum_{j=1}^m f(q_j) \quad (\text{E}[k])$$

will be denoted by  $(E^c[k])$  if  $(\text{E}[k])$  holds for all  $(p_1, \dots, p_n) \in \Gamma_n^c[k]$  and  $(q_1, \dots, q_m) \in \Gamma_m^c[k]$ , and the function  $f$  is defined on  $[0, 1]^k$  (closed domain case), and by  $(E^0[k])$  if  $(\text{E}[k])$  holds for all  $(p_1, \dots, p_n) \in \Gamma_n^0[k]$  and  $(q_1, \dots, q_m) \in \Gamma_m^0[k]$ , and  $f$  is defined on  $]0, 1[^k$  (open domain case). The solution of equation  $(E^c[1])$  is given by Losonczi and Maksa in [3], while equation  $(E^0[k])$  ( $k \in \mathbb{N}$ ) is solved by Ebanks, Sahoo, and Sander in [2].

**Theorem 1.1** (Losonczi, Maksa [3]). *Let  $n \geq 3$  and  $m \geq 3$  be fixed integers. A function  $f : [0, 1] \rightarrow \mathbb{R}$  satisfies  $(E^c[1])$  if, and only if, there exist additive functions  $A : \mathbb{R} \rightarrow \mathbb{R}$  and  $D : \mathbb{R} \rightarrow \mathbb{R}$ , a multiplicative function  $M : [0, 1] \rightarrow \mathbb{R}$ , and  $b \in \mathbb{R}$  such that  $D(\underline{1}) = 0$ ,  $A(\underline{1}) + nmb = (A(\underline{1}) + nb)(A(\underline{1}) + mb)$  and*

$$f(p) = A(p) + b, \quad p \in [0, 1]$$

or

$$f(p) = D(p) + M(p), \quad p \in [0, 1].$$

**Theorem 1.2** (Ebanks, Sahoo, Sander [2]). *Let  $k \geq 1$ ,  $n \geq 3$ , and  $m \geq 3$  be fixed integers. A function  $f : ]0, 1[^k \rightarrow \mathbb{R}$  satisfies  $(E^0[k])$  if, and only if, there exist additive functions  $A : \mathbb{R}^k \rightarrow \mathbb{R}$  and  $D : \mathbb{R}^k \rightarrow \mathbb{R}$ , a multiplicative function  $M : ]0, 1[^k \rightarrow \mathbb{R}$  and  $b \in \mathbb{R}$  such that  $D(\underline{1}) = 0$ ,  $A(\underline{1}) + nmb = (A(\underline{1}) + nb)(A(\underline{1}) + mb)$  and*

$$f(p) = A(p) + b, \quad p \in ]0, 1[^k$$

or

$$f(p) = D(p) + M(p), \quad p \in ]0, 1[^k.$$

The solution of equation  $(E^c[k])$  is not known if  $k \in \mathbb{N}$ ,  $k \geq 2$ . Our purpose is to solve equation  $(E^c[2])$ .

## 2. Preliminary results

**Lemma 2.1.** *Let  $k \geq 1$ ,  $n \geq 3$ , and  $m \geq 3$  be fixed integers. If the function  $f : [0, 1]^k \rightarrow \mathbb{R}$  satisfies  $(E^c[k])$  and  $A : \mathbb{R}^k \rightarrow \mathbb{R}$  is an additive function such that  $A(\underline{1}) = 0$  then the function  $g = f - A$  satisfies  $(E^c[k])$ , too.*

**Proof.**

$$\begin{aligned} \sum_{i=1}^n \sum_{j=1}^m g(p_i \bullet q_j) &= \sum_{i=1}^n \sum_{j=1}^m f(p_i \bullet q_j) - \sum_{i=1}^n \sum_{j=1}^m A(p_i \bullet q_j) = \\ &= \left( \sum_{i=1}^n f(p_i) - \sum_{i=1}^n A(p_i) \right) \left( \sum_{j=1}^m f(q_j) - \sum_{j=1}^m A(q_j) \right) = \sum_{i=1}^n g(p_i) \sum_{j=1}^m g(q_j). \end{aligned}$$

□

**Lemma 2.2.** *If  $A : \mathbb{R}^2 \rightarrow \mathbb{R}$  is additive,  $M : ]0, 1[^2 \rightarrow \mathbb{R}$  is multiplicative,  $H : ]0, 1[ \rightarrow \mathbb{R}$ , and  $M \begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} + H(x)$ ,  $\begin{pmatrix} x \\ y \end{pmatrix} \in ]0, 1[^2$  then*

$$M \begin{pmatrix} x \\ y \end{pmatrix} = \mu(x), \quad \begin{pmatrix} x \\ y \end{pmatrix} \in ]0, 1[^2,$$

where  $\mu : ]0, 1[ \rightarrow \mathbb{R}$  is a multiplicative function or

$$M \begin{pmatrix} x \\ y \end{pmatrix} = y, \quad \begin{pmatrix} x \\ y \end{pmatrix} \in ]0, 1[^2.$$

**Proof.** Let  $x, y, z \in ]0, 1[$ . Then  $A \begin{pmatrix} x \\ yz \end{pmatrix} + H(x) = M \begin{pmatrix} x \\ yz \end{pmatrix} = M \begin{pmatrix} \sqrt{x} \\ y \end{pmatrix} M \begin{pmatrix} \sqrt{x} \\ z \end{pmatrix} = \left( A \begin{pmatrix} \sqrt{x} \\ y \end{pmatrix} + H(\sqrt{x}) \right) \left( A \begin{pmatrix} \sqrt{x} \\ z \end{pmatrix} + H(\sqrt{x}) \right)$ . With fixed  $x$  and the notations  $a_1(t) = A \begin{pmatrix} x \\ t \end{pmatrix}$ ,  $t \in ]0, 1[$ ,  $a_2(t) = A \begin{pmatrix} \sqrt{x} \\ t \end{pmatrix}$ ,  $t \in ]0, 1[$  this implies that  $a_1(yz) + H(x) = (a_2(y) + H(\sqrt{x}))(a_2(z) + H(\sqrt{x}))$ , while with the substitutions  $y = z = \sqrt{t}$ ,  $a_1(t) + H(x) = (a_2(t) + H(\sqrt{x}))^2$ , that is,  $A \begin{pmatrix} 0 \\ t \end{pmatrix} = (a_2(t) + H(\sqrt{x}))^2 - A \begin{pmatrix} x \\ 0 \end{pmatrix} - H(x)$ ,  $t \in ]0, 1[$ . Since the function  $t \rightarrow A \begin{pmatrix} 0 \\ t \end{pmatrix}$  is additive and  $A \begin{pmatrix} 0 \\ t \end{pmatrix} \geq -A \begin{pmatrix} x \\ 0 \end{pmatrix} - H(x)$ ,  $t \in ]0, 1[$ , there exists  $c \in \mathbb{R}$  such that  $A \begin{pmatrix} 0 \\ t \end{pmatrix} = ct$  (see Aczél [1]), thus  $A \begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ 0 \end{pmatrix} +$

$cy, \begin{pmatrix} x \\ y \end{pmatrix} \in ]0, 1[^2$ , furthermore  $M \begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ 0 \end{pmatrix} + H(x) + cy, \begin{pmatrix} x \\ y \end{pmatrix} \in ]0, 1[^2$ . Let  $\mu(x) = A \begin{pmatrix} x \\ 0 \end{pmatrix} + H(x), x \in ]0, 1[$  and let  $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in ]0, 1[^2$ . Then  $cy_1y_2 + \mu(x_1x_2) = M \begin{pmatrix} x_1x_2 \\ y_1y_2 \end{pmatrix} = M \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} M \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = (cy_1 + \mu(x_1))(cy_2 + \mu(x_2))$ . Thus  $(c - c^2)y_1y_2 = \mu(x_1)\mu(x_2) - \mu(x_1x_2) + c(y_1\mu(x_2) + y_2\mu(x_1))$ . Taking here the limit  $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$  we have that  $\mu$  is multiplicative and

$$c(1 - c)y_1y_2 = c(y_1\mu(x_2) + y_2\mu(x_1)).$$

This implies that either  $c = 0$  and

$$M \begin{pmatrix} x \\ y \end{pmatrix} = \mu(x), \quad \begin{pmatrix} x \\ y \end{pmatrix} x \in ]0, 1[^2$$

or  $(1 - c)y_1y_2 = y_1\mu(x_2) + y_2\mu(x_1), \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in ]0, 1[^2$ . Since  $\mu$  is multiplicative, in this case we get that  $c = 1$  and  $A \begin{pmatrix} x \\ 0 \end{pmatrix} + H(x) = \mu(x) = 0, x \in ]0, 1[$ . Thus

$$M \begin{pmatrix} x \\ y \end{pmatrix} = y, \quad \begin{pmatrix} x \\ y \end{pmatrix} \in ]0, 1[^2.$$

□

**Lemma 2.3.** Suppose that  $3 \leq n \in \mathbb{N}, 3 \leq m \in \mathbb{N}, f : [0, 1]^2 \rightarrow \mathbb{R}$  satisfies equation  $(E^c[2])$  and

$$K = (m - 1)f(\underline{0}) + f(\underline{1}) = 1. \quad (2.1)$$

Then  $f(\underline{0}) = 0$  and  $f(\underline{1}) = 1$ .

**Proof.** Substituting  $P = (\underline{0}, \dots, \underline{0}, \underline{1}) \in \Gamma_m^c[2], Q = (\underline{0}, \dots, \underline{0}, \underline{1}) \in \Gamma_m^c[2]$  in  $(E^c[2])$ , by (2.1), we have  $(nm - 1)f(\underline{0}) + f(\underline{1}) = (n - 1)f(\underline{0}) + f(\underline{1})$  and, after some calculation, we get that  $n(m - 1)f(\underline{0}) = 0$ . This and (2.1) imply that  $f(\underline{0}) = 0$  and  $f(\underline{1}) = 1$ . □

### 3. The main result

**Theorem 3.1.** Let  $n \geq 3$  and  $m \geq 3$  be fixed integers. A function  $f : [0, 1]^2 \rightarrow \mathbb{R}$  satisfies  $(E^c[2])$  if, and only if, there exist additive functions  $A, D : \mathbb{R}^2 \rightarrow \mathbb{R}$ , a multiplicative function  $M : [0, 1]^2 \rightarrow \mathbb{R}$ , and  $b \in \mathbb{R}$  such that  $D(\underline{1}) = 0, A(\underline{1}) + nmb = (A(\underline{1}) + nb)(A(\underline{1}) + mb)$  and

$$f(p) = A(p) + b, \quad p \in [0, 1]^2$$

or

$$f(p) = D(p) + M(p), \quad p \in [0, 1]^2.$$

**Proof.** By Theorem 1.2, with  $k = 2$  we have that there exist additive functions  $A, D : \mathbb{R}^2 \rightarrow \mathbb{R}$ , a multiplicative function  $M : ]0, 1[^2 \rightarrow \mathbb{R}$  and  $b \in \mathbb{R}$  such that  $D(\underline{1}) = 0$ ,  $A(\underline{1}) + nmb = (A(\underline{1}) + nb)(A(\underline{1}) + mb)$  and

$$f(p) = A(p) + b, \quad p \in ]0, 1[^2$$

or

$$f(p) = D(p) + M(p), \quad p \in ]0, 1[^2.$$

We prove that, beside the conditions of Theorem 3.1,  $f$  has similar form with the same  $b \in \mathbb{R}$  and with the additive and multiplicative extensions of the functions  $A, D$ , and  $M$  onto the whole square  $[0, 1]^2$ , respectively. To have this result we will apply special substitutions in equation  $(E^c[2])$  to get information about the behavior of  $f$  on the boundary of  $[0, 1]^2$ .

CASE 1.  $f(p) = A(p) + b$ ,  $p \in ]0, 1[^2$  and  $A(\underline{1}) \neq 0$ .

SUBCASE 1.A.  $K \neq 1$  (see (2.1))

Substituting  $P = \begin{pmatrix} x & r & \cdots & r \\ 0 & u & \cdots & u \end{pmatrix} \in \Gamma_n^c[2]$ ,  $x \in ]0, 1[$ , and  $Q = (\underline{0}, \dots, \underline{0}, \underline{1}) \in \Gamma_m^c[2]$  in  $(E^c[2])$  we get that

$$\begin{aligned} n(m-1)f(\underline{0}) + f\left(\begin{pmatrix} x \\ 0 \end{pmatrix}\right) + A\left(\begin{pmatrix} 1-x \\ 1 \end{pmatrix}\right) + (n-1)b = \\ \left(f\left(\begin{pmatrix} x \\ 0 \end{pmatrix}\right) + A\left(\begin{pmatrix} 1-x \\ 1 \end{pmatrix}\right) + (n-1)b\right)K. \end{aligned}$$

Hence

$$f\left(\begin{pmatrix} x \\ 0 \end{pmatrix}\right) = A\left(\begin{pmatrix} x \\ 0 \end{pmatrix}\right) - A(\underline{1}) - (n-1)b + \frac{n(m-1)f(\underline{0})}{K-1} = A\left(\begin{pmatrix} x \\ 0 \end{pmatrix}\right) + b_{10}, \quad (3.1)$$

$x \in ]0, 1[$  for some  $b_{10} \in \mathbb{R}$ . A similar calculation shows that there exists  $b_{20} \in \mathbb{R}$  such that

$$f\left(\begin{pmatrix} 0 \\ y \end{pmatrix}\right) = A\left(\begin{pmatrix} 0 \\ y \end{pmatrix}\right) + b_{20}, \quad y \in ]0, 1[. \quad (3.2)$$

Substituting  $P = \begin{pmatrix} x & r & \cdots & r \\ 1 & 0 & \cdots & 0 \end{pmatrix} \in \Gamma_n^c[2]$ ,  $x \in ]0, 1[$ , and  $Q = (\underline{0}, \dots, \underline{0}, \underline{1}) \in \Gamma_m^c[2]$  in  $(E^c[2])$  we get that

$$\begin{aligned} n(m-1)f(\underline{0}) + f\left(\begin{pmatrix} x \\ 1 \end{pmatrix}\right) + A\left(\begin{pmatrix} 1-x \\ 0 \end{pmatrix}\right) + (n-1)b_{10} = \\ \left(f\left(\begin{pmatrix} x \\ 1 \end{pmatrix}\right) + A\left(\begin{pmatrix} 1-x \\ 0 \end{pmatrix}\right) + (n-1)b_{10}\right)K. \end{aligned}$$

Thus

$$f\left(\begin{pmatrix} x \\ 1 \end{pmatrix}\right) = A\left(\begin{pmatrix} x \\ 1 \end{pmatrix}\right) - A(\underline{1}) - (n-1)b_{10} + \frac{n(m-1)f(\underline{0})}{K-1} = A\left(\begin{pmatrix} x \\ 1 \end{pmatrix}\right) + b_{11}, \quad (3.3)$$

$x \in ]0, 1[$  for some  $b_{11} \in \mathbb{R}$ . A similar calculation shows that there exists  $b_{21} \in \mathbb{R}$  such that

$$f\left(\begin{array}{c} 1 \\ y \end{array}\right) = A\left(\begin{array}{c} 1 \\ y \end{array}\right) + b_{21}, \quad y \in ]0, 1[. \quad (3.4)$$

Now we show that  $b = b_{10} = b_{11} = b_{20} = b_{21}$ . Define the function  $g : [0, 1]^2 \rightarrow \mathbb{R}$  by  $g\left(\begin{array}{c} x \\ y \end{array}\right) = f\left(\begin{array}{c} x \\ y \end{array}\right) - \left(A\left(\begin{array}{c} x \\ y \end{array}\right) - A(\underline{1})x\right)$ . Then, by (3.1), (3.2), (3.3), and (3.4),  $g\left(\begin{array}{c} x \\ y \end{array}\right) = A(\underline{1})x + \delta$ ,  $\left(\begin{array}{c} x \\ y \end{array}\right) \in [0, 1]^2 \setminus \left\{\left(\begin{array}{c} 0 \\ 0 \end{array}\right), \left(\begin{array}{c} 0 \\ 1 \end{array}\right), \left(\begin{array}{c} 1 \\ 0 \end{array}\right), \left(\begin{array}{c} 1 \\ 1 \end{array}\right)\right\}$ , where  $\delta \in \{b, b_{10}, b_{11}, b_{20}, b_{21}\}$ , respectively. It follows from Lemma 2.1 that  $g$  satisfies equation  $(E^c[2])$ :

$$\sum_{i=1}^n \sum_{j=1}^m g(p_i \bullet q_j) = \sum_{i=1}^n g(p_i) \sum_{j=1}^m g(q_j) \quad (3.5)$$

Thus, with the substitutions,  $P = \left(\begin{array}{ccc} x_1 & \cdots & x_n \\ r & \cdots & r \end{array}\right) \in \Gamma_n^c[2]$ ,

$Q = \left(\begin{array}{ccc} y_1 & \cdots & y_m \\ s & \cdots & s \end{array}\right) \in \Gamma_m^c[2]$  in (3.5) we get that

$$\sum_{i=1}^n \sum_{j=1}^m g\left(\begin{array}{c} x_i y_j \\ r s \end{array}\right) = \sum_{i=1}^n g\left(\begin{array}{c} x_i \\ r \end{array}\right) \sum_{j=1}^m g\left(\begin{array}{c} y_j \\ s \end{array}\right),$$

$(x_1, \dots, x_n) \in \Gamma_n^c[1], (y_1, \dots, y_m) \in \Gamma_m^c[1]$ . Let  $\zeta \in ]0, 1[$  be fixed and  $G_\zeta(x) = g(x, \zeta)$ ,  $x \in [0, 1]$ . Since  $g$  does not depend on its second variable if it is from  $]0, 1[$ ,  $G_\zeta$  satisfies equation  $(E^c[1])$ . Concerning  $G_\zeta(x) = A(\underline{1})x + b$ ,  $x \in ]0, 1[$  and  $A(\underline{1}) \neq 0$ , by Theorem 1.1, we have that  $G_\zeta(x) = A(\underline{1})x + b$ ,  $x \in [0, 1]$ , that is,  $b = b_{20} = b_{21}$ . In a similar way we can get that  $b = b_{10} = b_{11}$ , that is,

$$g\left(\begin{array}{c} x \\ y \end{array}\right) = A(\underline{1})x + b, \quad \left(\begin{array}{c} x \\ y \end{array}\right) \in [0, 1]^2 \setminus \left\{\left(\begin{array}{c} 0 \\ 0 \end{array}\right), \left(\begin{array}{c} 0 \\ 1 \end{array}\right), \left(\begin{array}{c} 1 \\ 0 \end{array}\right), \left(\begin{array}{c} 1 \\ 1 \end{array}\right)\right\}. \quad (3.6)$$

Now we prove that (3.6) holds on  $[0, 1]^2$ . Let  $G_0(x) = g\left(\begin{array}{c} x \\ 0 \end{array}\right)$ ,  $x \in [0, 1]$ .  $G_0(x) = A(\underline{1})x + b$ ,  $x \in ]0, 1[$ . Thus  $G_0$  satisfies  $(E^0[2])$ . We show that  $G_0$  satisfies  $(E^c[2])$ , too. Let  $(p_1, \dots, p_n) = \left(\begin{array}{ccc} x_1 & \cdots & x_{n-1} & x_n \\ 0 & \cdots & 0 & 1 \end{array}\right) \in \Gamma_n^c[2]$ ,

$(q_1, \dots, q_m) = \left(\begin{array}{ccc} y_1 & \cdots & y_{m-1} & y_m \\ 0 & \cdots & 0 & 1 \end{array}\right) \in \Gamma_m^c[2]$ ,  $x_1, \dots, x_n, y_1, \dots, y_m \in [0, 1[$ .

Since  $g\left(\begin{array}{c} t \\ 0 \end{array}\right) = g\left(\begin{array}{c} t \\ 1 \end{array}\right)$ ,  $t \in ]0, 1[$  we have that

$$\sum_{i=1}^n \sum_{j=1}^m G_0(x_i y_j) = \sum_{i=1}^n \sum_{j=1}^m g(p_i \bullet q_j) =$$

$$\sum_{i=1}^n g(p_i) \sum_{j=1}^m g(q_j) = \sum_{i=1}^n G_0(x_i) \sum_{j=1}^m G_0(q_j). \quad (3.7)$$

Substituting  $x_1 = \dots = x_{n-2} = 0, x_{n-1} = x_n = \frac{1}{2}$ ,  $y_1 = \dots = y_m = \frac{1}{m}$  in (3.7) and using the equalities  $G_0(x) = A(\underline{1})x + b$ ,  $x \in ]0, 1[$  and  $A(\underline{1}) + nmb = (A(\underline{1}) + nb)(A(\underline{1}) + mb)$  we get that

$$(G_0(0) - b)(nm - 2m - nA(\underline{1}) - nmb + 2A(\underline{1}) + 2mb) = 0.$$

An easy calculation shows that the condition  $A(\underline{1}) \neq 0$  implies that  $(nm - 2m - nA(\underline{1}) - nmb + 2A(\underline{1}) + 2mb) \neq 0$ , that is  $g(\underline{0}) = G_0(0) = b$ .

The substitutions  $P = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & r & \dots & r \end{pmatrix} \in \Gamma_n^c[2]$ ,  $Q = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & s & \dots & s \end{pmatrix} \in \Gamma_m^c[2]$  and  $P = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & u & \dots & u \end{pmatrix} \in \Gamma_n^c[2]$ ,  $Q = \begin{pmatrix} y_1 & \dots & y_m \\ v & \dots & v \end{pmatrix} \in \Gamma_m^c[2]$  in (3.5), using  $G_0(0) = b$ , imply that the function  $G_0$  satisfies equation  $(E^c[1])$  also in the remaining cases  $x_1 = 1, x_2 = \dots = x_n = 0, y_1 = 1, y_2 = \dots = y_m = 0$  and  $x_1 = 1, x_2 = \dots = x_n = 0, (y_1, \dots, y_m) \in \Gamma_m^c[1]$ . Thus, by Theorem 1.1,  $G_0(x) = A(\underline{1})x + b$ ,  $x \in [0, 1]$ , that is,  $g\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = G_0(1) = A(\underline{1}) + b$ . In a similar way we can get that  $g\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) = A(\underline{1}) + b$ . Finally the following calculation proves that

$g(\underline{1}) = A(\underline{1}) + b$ . Substituting  $P = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 1 \end{pmatrix} \in \Gamma_n^c[2]$ ,  $Q = (\underline{1}, \underline{0}, \dots, \underline{0}) \in \Gamma_m^c[2]$  in (3.5) we have that  $(A(\underline{1}) + nb)(g(\underline{1}) - A(\underline{1}) - b) = 0$ . It is easy to see that the condition  $A(\underline{1}) \neq 0$  implies that  $A(\underline{1}) + nb \neq 0$  thus  $g(\underline{1}) = A(\underline{1}) + b$ .

SUBCASE 1.B.  $K = 1$  (see (2.1))

In this case, by Lemma 2.3,  $f(\underline{0}) = 0$  and  $f(\underline{1}) = 1$ . Substituting

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \dots & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \dots & 0 \end{pmatrix} \in \Gamma_n^c[2], Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \dots & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \dots & 0 \end{pmatrix} \in \Gamma_m^c[2],$$

$$P = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \dots & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \dots & 0 \end{pmatrix} \in \Gamma_n^c[2], Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \dots & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \dots & 0 \end{pmatrix} \in \Gamma_m^c[2],$$

$$P = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \dots & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \dots & 0 \end{pmatrix} \in \Gamma_n^c[2], Q = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \dots & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \dots & 0 \end{pmatrix} \in \Gamma_m^c[2] \text{ in } (E^c[2]) \text{ we get the following system of equations.}$$

$$\begin{aligned} I. \quad & A(\underline{1}) + 4b = (A(\underline{1}) + 2b)^2 \\ II. \quad & A(\underline{1}) + 6b = (A(\underline{1}) + 2b)(A(\underline{1}) + 3b) \\ III. \quad & A(\underline{1}) + 9b = (A(\underline{1}) + 3b)^2. \end{aligned}$$

This and the condition  $A(\underline{1}) \neq 0$  imply that  $b = 0$ , furthermore  $A(\underline{1}) = 1$ , that is,  $f(\underline{0}) = 0$  and  $f(\underline{1}) = 1$ . Substituting  $P = \begin{pmatrix} 1 & 0 & 0 \dots & 0 \\ 0 & 1 & 0 \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$ ,  $Q = \begin{pmatrix} 1 & 0 & 0 \dots & 0 \\ 0 & 1 & 0 \dots & 0 \end{pmatrix} \in \Gamma_m^c[2]$  in  $(E^c[2])$  we get that  $f\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) + f\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) =$

$\left(f\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) + f\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\right)^2$  thus  $f\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) + f\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \in \{0, 1\}$ , while with the substitutions  $P = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_n^0[2]$ ,  $Q = (q_1, \dots, q_m) \in \Gamma_m^0[2]$  in  $(E^c[2])$  we get that

$$\sum_{j=1}^m f\left(\begin{smallmatrix} q_{j1} \\ 0 \end{smallmatrix}\right) + \sum_{j=1}^m f\left(\begin{smallmatrix} 0 \\ q_{j2} \end{smallmatrix}\right) = \left(f\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) + f\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\right) \sum_{j=1}^m f(q_j). \quad (3.8)$$

If  $f\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) + f\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) = 0$  then, with fixed  $Q = (q_{12}, \dots, q_{m2})$ , (3.8) goes over into  $\sum_{j=1}^m f\left(\begin{smallmatrix} q_{j1} \\ 0 \end{smallmatrix}\right) = c$ ,  $(q_{11}, \dots, q_{m1}) \in \Gamma_m^0[1]$  with some  $c \in \mathbb{R}$ , so, by Theorem 1.2, there exist additive function  $a_{10} : \mathbb{R} \rightarrow \mathbb{R}$  and  $b_{10} \in \mathbb{R}$  such that

$$f\left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right) = a_{10}(x) + b_{10}, \quad x \in ]0, 1[. \quad (3.9)$$

In a similar way we can prove that there exist an additive function  $a_{20} : \mathbb{R} \rightarrow \mathbb{R}$  and  $b_{20} \in \mathbb{R}$  such that

$$f\left(\begin{smallmatrix} 0 \\ y \end{smallmatrix}\right) = a_{20}(y) + b_{20}, \quad y \in ]0, 1[. \quad (3.10)$$

If  $f\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) + f\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) = 1$  then (3.8) goes over into  $\sum_{j=1}^m \left[ f\left(\begin{smallmatrix} q_{j1} \\ q_{j2} \end{smallmatrix}\right) - f\left(\begin{smallmatrix} q_{j1} \\ 0 \end{smallmatrix}\right) - f\left(\begin{smallmatrix} 0 \\ q_{j2} \end{smallmatrix}\right) \right] = 0$ ,  $(q_1, \dots, q_m) \in \Gamma_m^0[2]$ . Thus there exist an additive function  $A_0 : \mathbb{R}^2 \rightarrow \mathbb{R}$  and  $b_0 \in \mathbb{R}$  such that

$$f\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) - f\left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right) - f\left(\begin{smallmatrix} 0 \\ y \end{smallmatrix}\right) = A_0\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) + b_0, \quad \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in ]0, 1[^2. \quad (3.11)$$

With the functions  $a_{10}(x) = (A - A_0)\left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right)$ ,  $x \in ]0, 1[$  and  $a_{20}(y) = (A - A_0)\left(\begin{smallmatrix} 0 \\ y \end{smallmatrix}\right)$ ,  $y \in ]0, 1[$  we have that

$$f\left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right) = a_{10}(x) + \left(a_{20}(y) - f\left(\begin{smallmatrix} 0 \\ y \end{smallmatrix}\right) + b_0\right), \quad x \in ]0, 1[$$

and

$$f\left(\begin{smallmatrix} 0 \\ y \end{smallmatrix}\right) = a_{20}(y) + \left(a_{10}(x) - f\left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right) + b_0\right), \quad y \in ]0, 1[.$$

With fixed  $x$  and  $y$ , we obtain again that (3.9) and (3.10) hold with some  $b_{10} \in \mathbb{R}$  and  $b_{20} \in \mathbb{R}$ , respectively.



Substituting  $P = \begin{pmatrix} x & r & \cdots & r \\ 1 & 0 & \cdots & 0 \end{pmatrix} \in \Gamma_n^c[2]$ ,  $x \in ]0, 1[$ ,  $Q = (q_1, \dots, q_m) \in \Gamma_m^0[2]$  in  $(E^c[2])$ , after some calculation, we get that

$$f \begin{pmatrix} x \\ 1 \end{pmatrix} = A \begin{pmatrix} x \\ 1 \end{pmatrix}, \quad x \in ]0, 1[. \quad (3.12)$$

In a similar way we have that

$$f \begin{pmatrix} 1 \\ y \end{pmatrix} = A \begin{pmatrix} 1 \\ y \end{pmatrix}, \quad y \in ]0, 1[. \quad (3.13)$$

Substituting  $P = \begin{pmatrix} x & r & \cdots & r \\ 0 & u & \cdots & u \end{pmatrix} \in \Gamma_m^0[2]$ ,  $x \in ]0, 1[$ ,  $Q = \begin{pmatrix} s & \cdots & s \\ s & \cdots & s \end{pmatrix}$  in  $(E^c[2])$ , after some calculation, we have that  $b_{10} = 0$  and, in a similar way, we get that  $b_{20} = 0$ . Substituting  $P = \begin{pmatrix} x & 1-x & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{pmatrix} \in \Gamma_m^c[2]$ ,  $x \in ]0, 1[$ ,  $Q = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ y & 1-y & 0 & \cdots & 0 \end{pmatrix} \in \Gamma_m^c[2]$ ,  $y \in ]0, 1[$  in  $(E^c[2])$ , after some calculation, we have that

$$\left( a_{10}(x) - A \begin{pmatrix} x \\ 0 \end{pmatrix} \right) + \left( a_{20}(y) - A \begin{pmatrix} 0 \\ y \end{pmatrix} - 1 \right) = a_{20}(y) - A \begin{pmatrix} 0 \\ y \end{pmatrix}.$$

This implies that either

$$a_{10}(x) = A \begin{pmatrix} x \\ 0 \end{pmatrix}, \quad x \in ]0, 1[ \quad (3.14)$$

and

$$a_{20}(y) = A \begin{pmatrix} 0 \\ y \end{pmatrix}, \quad y \in ]0, 1[, \quad (3.15)$$

or none of these equations holds. It is easy to see that the later case is not possible.

Thus (3.14) and (3.15) hold. Finally with the substitutions  $P = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & r & \cdots & r \end{pmatrix} \in \Gamma_m^c[2]$ ,  $Q = \begin{pmatrix} s & \cdots & s \\ s & \cdots & s \end{pmatrix}$  in  $(E^c[2])$ , after some calculation, we have that  $f \begin{pmatrix} 1 \\ 0 \end{pmatrix} = A \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ . In a similar way we get that  $f \begin{pmatrix} 0 \\ 1 \end{pmatrix} = A \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ .

CASE 2.

$$f(x) = A(x) + b, \quad x \in ]0, 1[^2, \quad A(1) = 0 \quad (3.16)$$

or

$$f(x) = D(x) + M(x), \quad x \in ]0, 1[^2, \quad D(1) = 0. \quad (3.17)$$

Define the function  $g$  by  $f - A$  if (3.16) holds and by  $f - D$  if (3.17) holds. It is easy to see that we have to investigate the following three subcases.

SUBCASE 2.A.  $g(x) = 0$ ,  $x \in ]0, 1[^2$ , when

$$f(x) = A(x) + b, \quad b = 0, \quad x \in ]0, 1[^2$$

or

$$f(x) = D(x) + M(x), \quad M(x) = 0, \quad x \in ]0, 1[^2,$$

SUBCASE 2.B.  $g(x) = 1, x \in ]0, 1[^2$ , when

$$f(x) = A(x) + b, \quad b = 1, \quad x \in ]0, 1[^2$$

or

$$f(x) = D(x) + M(x) \quad M(x) = 1, \quad x \in ]0, 1[^2,$$

SUBCASE 2.C.  $g(x) = 0, x \in ]0, 1[^2, \quad M \neq 0, M \neq 1$ , when

$$f(x) = D(x) + M(x), \quad x \in ]0, 1[^2, \quad M \neq 0, M \neq 1.$$

By Lemma 2.1, the function  $g$  satisfies  $(E^c[2])$ :

$$\sum_{i=1}^n \sum_{j=1}^m g(p_i \bullet q_j) = \sum_{i=1}^n g(p_i) \sum_{j=1}^m g(q_j) \quad (3.18)$$

SUBCASE 2.A.  $g(x) = 0, x \in ]0, 1[^2$ . With the substitutions

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \dots & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \dots & 0 \end{pmatrix} \in \Gamma_n^c[2], \quad Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \dots & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \dots & 0 \end{pmatrix} \in \Gamma_m^c[2],$$

$$P = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \dots & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \dots & 0 \end{pmatrix} \in \Gamma_n^c[2], \quad Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \dots & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \dots & 0 \end{pmatrix} \in \Gamma_m^c[2]$$

in (3.18), after some calculation, we have that  $g(0) = 0$ . With the substitutions

$$P = \begin{pmatrix} x & r & \dots & r \\ 0 & u & \dots & u \end{pmatrix} \in \Gamma_n^c[2], \quad x \in ]0, 1[, \quad Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & \dots & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & \dots & 0 \end{pmatrix} \in \Gamma_m^c[2] \text{ in}$$

(3.18) we get that

$$g \begin{pmatrix} x \\ 0 \end{pmatrix} = 0, \quad x \in \left]0, \frac{1}{2}\right[, \quad (3.19)$$

while with the substitutions  $P = \begin{pmatrix} x & r & \dots & r \\ 0 & u & \dots & u \end{pmatrix} \in \Gamma_n^c[2], x \in ]0, 1[, \quad Q =$

$(q_1, \dots, q_m) \in \Gamma_m^0[2]$  in (3.18) we have that

$$\sum_{j=1}^n g \begin{pmatrix} xq_{j1} \\ 0 \end{pmatrix} = 0, \quad (q_{11}, \dots, q_{m1}) \in \Gamma_m^0[1].$$

Hence there exists additive function  $a_x : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$g \begin{pmatrix} q \\ 0 \end{pmatrix} = a_x \left( \frac{x}{q} \right) - \frac{a_x(1)}{n}, \quad q \in ]0, x[, \quad (3.20)$$

where  $x$  is an arbitrary fixed element of  $]0, 1[$ . It follows from (3.19) and (3.20) that

$$g \begin{pmatrix} x \\ 0 \end{pmatrix} = 0, \quad x \in ]0, 1[. \quad (3.21)$$

In a similar way we get that

$$g\left(\begin{smallmatrix} 0 \\ y \end{smallmatrix}\right) = 0, \quad y \in ]0, 1[. \quad (3.22)$$

It is easy to see that

$$\left(g\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), g\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right), g\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)\right) \in \{(0, 0, 0), (0, 0, 1), (1, 0, 1), (0, 1, 1)\}. \quad (3.23)$$

Indeed, the substitutions

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \end{pmatrix} \in \Gamma_n^c[2], \quad Q = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \end{pmatrix} \in \Gamma_m^c[2],$$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \end{pmatrix} \in \Gamma_n^c[2], \quad Q = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \in \Gamma_m^c[2],$$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \end{pmatrix} \in \Gamma_n^c[2], \quad Q = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{pmatrix} \in \Gamma_m^c[2], \text{ and}$$

$$P = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{pmatrix} \in \Gamma_n^c[2], \quad Q = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{pmatrix} \in \Gamma_m^c[2]$$

in (3.18) imply that

$$g\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = \left(g\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\right)^2 \text{ thus } g\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \in \{0, 1\},$$

$$g\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) g\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) = 0,$$

$$g\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = g\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) g\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) \text{ thus if } g\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = 1 \text{ then } g\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) = 1, \text{ and}$$

$$g\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) = \left(g\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)\right)^2 \text{ thus } g\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) \in \{0, 1\},$$

respectively. In a similar way we get that  $g\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \in \{0, 1\}$ , and if  $g\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) = 1$

then  $g\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) = 1$ , respectively, that is, (3.23) holds.

Now we show that the statement of our theorem holds in each case given by (3.23).

The substitutions  $P = \begin{pmatrix} x & r & \cdots & r \\ 1 & 0 & \cdots & 0 \end{pmatrix} \in \Gamma_n^c[2]$ ,  $x \in ]0, 1[$ ,  $Q = \begin{pmatrix} 0 & s & \cdots & s \\ 1 & 0 & \cdots & 0 \end{pmatrix}$

$\in \Gamma_m^c[2]$  in (3.18) imply that  $g\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) = g\left(\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right) g\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$  thus, if  $g\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) = 1$ ,

then  $g\left(\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right) = 1$ ,  $x \in [0, 1]$ . In a similar way we have that, if  $g\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = 1$ , then

$g\left(\begin{smallmatrix} 1 \\ y \end{smallmatrix}\right) = 1$ ,  $y \in [0, 1]$ . The substitutions  $P = \begin{pmatrix} x & r & \cdots & r \\ 1 & 0 & \cdots & 0 \end{pmatrix} \in \Gamma_n^c[2]$ ,  $x \in$

$]0, 1[$ ,  $Q = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ y & s & \cdots & s \end{pmatrix} \in \Gamma_m^c[2]$  in (3.18) imply that  $g\left(\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right) g\left(\begin{smallmatrix} 1 \\ y \end{smallmatrix}\right) =$

0. Thus  $g\left(\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right) = 0$ ,  $x \in [0, 1]$  or  $g\left(\begin{smallmatrix} 1 \\ y \end{smallmatrix}\right) = 0$ ,  $y \in [0, 1]$ . In the remaining

case  $g\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = g\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) = 0$ , substitute  $P = \begin{pmatrix} x & r & \cdots & r \\ 1 & 0 & \cdots & 0 \end{pmatrix} \in \Gamma_n^c[2]$ ,  $x \in ]0, 1[$ ,  $Q = \begin{pmatrix} y & s & \cdots & s \\ 1 & 0 & \cdots & 0 \end{pmatrix} \in \Gamma_m^c[2]$ ,  $y \in ]0, 1[$  in (3.18). Then we have that  $g\left(\begin{smallmatrix} xy \\ 1 \end{smallmatrix}\right) = g\left(\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right)g\left(\begin{smallmatrix} y \\ 1 \end{smallmatrix}\right)$ ,  $x, y \in ]0, 1[$ , that is, the function  $\mu_1(x) = g\left(\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right)$ ,  $x \in ]0, 1[$  is multiplicative. In a similar way we can see that the function  $\mu_2(y) = g\left(\begin{smallmatrix} 1 \\ y \end{smallmatrix}\right)$ ,  $y \in ]0, 1[$  is multiplicative, too.

SUBCASE 2.B.  $g(x) = 1$ ,  $x \in ]0, 1[^2$ . The substitutions  $P = \begin{pmatrix} x & r & \cdots & r \\ 0 & u & \cdots & u \end{pmatrix} \in \Gamma_n^c[2]$ ,  $x \in ]0, 1[$ ,  $Q = (q_1, \dots, q_m) \in \Gamma_m^0[2]$  in (3.18), imply that

$$\sum_{j=1}^m \left[ g\left(\begin{smallmatrix} xq_{j1} \\ 0 \end{smallmatrix}\right) - g\left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right) \right] = 0, \quad (q_{11}, \dots, q_{1m}) \in \Gamma_m^0[1].$$

Thus there exists an additive function  $a_x : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$g\left(\begin{smallmatrix} q \\ 0 \end{smallmatrix}\right) = a_x\left(\frac{x}{q}\right) + g\left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right) - \frac{a_x(1)}{n}, \quad q \in ]0, x[,$$

where  $x$  is an arbitrary fixed element of  $]0, 1[$ . This implies that there exist additive function  $a_1 : \mathbb{R} \rightarrow \mathbb{R}$  and  $c_1 \in \mathbb{R}$  such that

$$g\left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right) = a_1(x) + c_1, \quad x \in ]0, 1[.$$

In a similar way we get that there exist additive function  $a_2 : \mathbb{R} \rightarrow \mathbb{R}$  and  $c_2 \in \mathbb{R}$  such that

$$g\left(\begin{smallmatrix} 0 \\ y \end{smallmatrix}\right) = a_2(y) + c_2, \quad y \in ]0, 1[.$$

With the substitutions  $P = \begin{pmatrix} x & r & \cdots & r \\ 1 & 0 & \cdots & 0 \end{pmatrix} \in \Gamma_n^c[2]$ ,  $x \in ]0, 1[$ ,  $Q = (q_1, \dots, q_m) \in \Gamma_m^0[2]$  in (3.18) we get that

$$g\left(\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right) = \frac{m-1}{m}a_1(x-1) + 1, \quad x \in ]0, 1[.$$

Similarly we have that

$$g\left(\begin{smallmatrix} 1 \\ y \end{smallmatrix}\right) = \frac{m-1}{m}a_2(y-1) + 1, \quad y \in ]0, 1[.$$

With the substitutions  $P = \begin{pmatrix} 0 & r & \cdots & r \\ 0 & u & \cdots & u \end{pmatrix} \in \Gamma_n^c[2]$ ,  $Q = \begin{pmatrix} 0 & s & \cdots & s \\ 0 & v & \cdots & v \end{pmatrix} \in \Gamma_m^c[2]$  in (3.18), after some calculation, we get that  $(g(\underline{0}))^2 = g(\underline{0})$ , so  $g(\underline{0}) \in \{0, 1\}$ .

If  $g(\underline{0}) = 0$  then, with the substitutions  $P = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{pmatrix} \in \Gamma_n^c[2]$ ,  $Q = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{pmatrix} \in \Gamma_m^c[2]$  in (3.18), we get that  $(g(\underline{1}))^2 = g(\underline{1})$ , so  $g(\underline{1}) \in \{0, 1\}$ .

Furthermore, with the substitutions  $P = \begin{pmatrix} x & 1-x & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{pmatrix} \in \Gamma_n^c[2]$ ,  $x \in ]0, 1[$ ,  $Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \end{pmatrix} \in \Gamma_m^c[2]$  in (3.18), we get that

$$a_1(x) = 0, \quad x \in ]0, 1[.$$

In a similar way we obtain that

$$a_2(y) = 0, \quad y \in ]0, 1[.$$

With the substitutions

$$P = \begin{pmatrix} x & r & \cdots & r \\ 0 & u & \cdots & u \end{pmatrix} \in \Gamma_n^c[2], \quad Q = \begin{pmatrix} y & s & \cdots & s \\ 0 & v & \cdots & v \end{pmatrix} \in \Gamma_m^c[2], \quad x, y \in ]0, 1[,$$

$$P = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{pmatrix} \in \Gamma_n^c[2], \quad Q = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{pmatrix} \in \Gamma_m^c[2],$$

$$P = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & r & \cdots & r \end{pmatrix} \in \Gamma_n^c[2] \quad Q = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & s & \cdots & s \end{pmatrix} \in \Gamma_m^c[2], \text{ and}$$

$$P = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{pmatrix} \in \Gamma_n^c[2], \quad Q = (q_1, \dots, q_m) \in \Gamma_m^0[2]$$

in (3.18), after some calculation, we get that

$c_1 = 0$  (a similar calculation shows that  $c_2 = 0$ ),

$$g\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) + g\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \left(g\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) + g\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)\right)^2, \text{ that is, } g\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) + g\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \in \{0, 1\},$$

$$g\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) g\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = 0, \text{ and}$$

$g(\underline{1}) = 1$ , respectively.

If  $g(\underline{0}) = 1$  then, with the substitutions  $P = \begin{pmatrix} x & 1-x & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{pmatrix} \in \Gamma_n^c[2]$ ,

$x \in ]0, 1[$ ,  $Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \end{pmatrix} \in \Gamma_m^c[2]$  in (3.18), after some calculation, we get that  $c_1 = 1$ . In a similar way we have that  $c_2 = 1$ . The substitutions

$$P = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{pmatrix} \in \Gamma_n^c[2], \quad Q = (q_1, \dots, q_m) \in \Gamma_m^0[2] \text{ in (3.18) imply that}$$

$g(\underline{1}) = 1$ . With the substitutions  $P = \begin{pmatrix} x & r & \cdots & r \\ 1 & 0 & \cdots & 0 \end{pmatrix} \in \Gamma_n^c[2]$ ,  $x \in ]0, 1[$ ,

$$Q = \begin{pmatrix} y & s & \cdots & s \\ 1 & 0 & \cdots & 0 \end{pmatrix} \in \Gamma_m^c[2], \quad y \in ]0, 1[ \text{ in (3.18) we get that}$$

$$\frac{1}{m^2} a_1(x) a_1(y) = a_1(x) \left(1 + \frac{a_1(1)}{m}\right) +$$

$$a_1(y) \left( \frac{n}{m} + \frac{a_1(1)}{m} \right) - \frac{a_1(xy)}{m} + a_1(1)(1 - n - m - a_1(1))$$

From this, with  $y = \frac{1}{2}$ , after some calculation, we get that

$$a_1(x) = \frac{ma_1(1)}{a_1(1) + m} \left( n + a_1(1) + \frac{2m^2 - 1}{2m - 1} \right). \quad (3.24)$$

Since  $a_1$  is additive and the right hand side of (3.24) does not depend on  $x$  we have that

$$a_1(x) = 0, \quad x \in ]0, 1[.$$

In a similar way, we have that

$$a_2(y) = 0, \quad y \in ]0, 1[.$$

With the substitutions  $P = \begin{pmatrix} 0 & r & \cdots & r \\ 1 & 0 & \cdots & 0 \end{pmatrix} \in \Gamma_n^c[2]$ ,  $Q = (q_1, \dots, q_m) \in \Gamma_m^0[2]$  in (3.18) we get that  $g \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1$ . In a similar way, we get that  $g \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ . Thus

$$g(x) = 1, \quad x \in [0, 1]^2.$$

SUBCASE 2.C.  $g(x) = M(x)$ ,  $x \in ]0, 1[^2$ , where  $M : ]0, 1[^2 \rightarrow \mathbb{R}$  is a multiplicative function which is different from the following four functions:  $\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow 0, \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow 1, \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow x, \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow y, \begin{pmatrix} x \\ y \end{pmatrix} \in ]0, 1[^2$ . It is easy to check that this condition implies that there does not exist  $c \in \mathbb{R}$  such that  $\sum_{j=1}^n M(q_j) = c$  for all  $Q = (q_1, \dots, q_m) \in \Gamma_m^0[2]$ .

With the substitutions  $P = \begin{pmatrix} 0 & r & \cdots & r \\ 0 & u & \cdots & u \end{pmatrix} \in \Gamma_n^c[2]$ ,  $Q = (q_1, \dots, q_m) \in \Gamma_m^0[2]$  in (3.18) we get that

$$g(\underline{0}) \left( \sum_{j=1}^n M(q_j) - m \right) = 0.$$

Since there exists  $Q^0 \in \Gamma_m^0[2]$  such that  $\sum_{j=1}^n M(q_j^0) \neq m$  thus  $g(\underline{0}) = 0$ . With the substitutions  $P = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{pmatrix} \in \Gamma_n^c[2]$ ,  $Q = (q_1, \dots, q_m) \in \Gamma_m^0[2]$  in (3.18) we get that  $(g(\underline{1}) - 1) \sum_{j=1}^n M(q_j) = 0$ . Since there exists  $Q^0 \in \Gamma_m^0[2]$  such that  $\sum_{j=1}^n M(q_j^0) \neq 0$  thus  $g(\underline{1}) = 1$ . The substitutions  $P = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{pmatrix} \in \Gamma_n^c[2]$ ,  $Q = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{pmatrix} \in \Gamma_m^c[2]$  in (3.18) imply that  $g \begin{pmatrix} 1 \\ 0 \end{pmatrix} + g \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \left( g \begin{pmatrix} 1 \\ 0 \end{pmatrix} + g \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)^2$ , that is,  $g \begin{pmatrix} 1 \\ 0 \end{pmatrix} + g \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \{0, 1\}$ . The following

calculation shows that, if there exists  $x_0 \in ]0, 1[$  such that  $g\left(\begin{smallmatrix} x_0 \\ 0 \end{smallmatrix}\right) \neq 0$ , then there exists a multiplicative function  $\mu : ]0, 1[ \rightarrow \mathbb{R}$  such that  $M\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \mu(x)$ ,  $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in ]0, 1[^2$ . The substitutions  $P = \begin{pmatrix} x_0 & r & \cdots & r \\ 0 & u & \cdots & u \end{pmatrix} \in \Gamma_n^c[2]$ ,  $x_0 \in ]0, 1[$ ,  $Q = (q_1, \dots, q_m) \in \Gamma_m^0[2]$  in (3.18), imply that

$$\sum_{j=1}^m \left[ g\left(\begin{smallmatrix} x_0 q_{j1} \\ 0 \end{smallmatrix}\right) - g\left(\begin{smallmatrix} x_0 \\ 0 \end{smallmatrix}\right) M(q_j) \right] = 0, \quad Q = (q_1, \dots, q_m) \in \Gamma_m^0[2].$$

Thus there exists an additive function  $A_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$  such that

$$M\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \frac{-1}{g\left(\begin{smallmatrix} x_0 \\ 0 \end{smallmatrix}\right)} A_1\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) + \frac{1}{g\left(\begin{smallmatrix} x_0 \\ 0 \end{smallmatrix}\right)} \left[ g\left(\begin{smallmatrix} x_0 x \\ 0 \end{smallmatrix}\right) - \frac{A(1)}{m} \right].$$

Hence there exist an additive function  $A : \mathbb{R}^2 \rightarrow \mathbb{R}$  and a function  $H : ]0, 1[ \rightarrow \mathbb{R}$  such that

$$M\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = A\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) + H(x), \quad \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in ]0, 1[^2.$$

Since the case  $M\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = y$ ,  $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in ]0, 1[^2$  is excluded, by Lemma 2.2, there exists multiplicative function  $\mu : ]0, 1[ \rightarrow \mathbb{R}$  such that  $M\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \mu(x)$ ,  $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in ]0, 1[^2$ . In a similar way we can prove that, if there exists  $y_0 \in ]0, 1[$  such that  $g\left(\begin{smallmatrix} 0 \\ y_0 \end{smallmatrix}\right) \neq 0$ , then there exists a multiplicative function  $\mu : ]0, 1[ \rightarrow \mathbb{R}$  such that  $M\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \mu(y)$ ,  $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in ]0, 1[^2$ .

Now we show that  $g\left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right) = 0$ ,  $x \in ]0, 1[$  or  $g\left(\begin{smallmatrix} 0 \\ y \end{smallmatrix}\right) = 0$ ,  $y \in ]0, 1[$ . Indeed, suppose that there exist  $x_0 \in ]0, 1[$  and  $y_0 \in ]0, 1[$  such that  $g\left(\begin{smallmatrix} x_0 \\ 0 \end{smallmatrix}\right) \neq 0$  and  $g\left(\begin{smallmatrix} 0 \\ y_0 \end{smallmatrix}\right) \neq 0$ . Then there exist multiplicative functions  $\mu_1 : ]0, 1[ \rightarrow \mathbb{R}$  and  $\mu_2 : ]0, 1[ \rightarrow \mathbb{R}$  such that  $M\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \mu_1(x) = \mu_2(y)$ ,  $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in ]0, 1[^2$ . This implies that  $M\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = 0$ ,  $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in ]0, 1[^2$  or  $M\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = 1$ ,  $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in ]0, 1[^2$ , which are excluded in this case.

If  $g\left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right) = 0$ ,  $x \in ]0, 1[$  and  $g\left(\begin{smallmatrix} 0 \\ y \end{smallmatrix}\right) = 0$ ,  $y \in ]0, 1[$  then substitute

$P = \begin{pmatrix} 0 & r & \cdots & r \\ 1 & 0 & \cdots & 0 \end{pmatrix} \in \Gamma_n^c[2]$ ,  $Q = (q_1, \dots, q_m) \in \Gamma_m^0[2]$  in (3.18). Thus we get

that

$$g \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sum_{j=1}^m M(q_j) = 0.$$

Since there exists  $Q^0 \in \Gamma_m^0[2]$  such that  $\sum_{j=1}^m M(q_j^0) \neq 0$  therefore  $g \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$ . In a similar way we have that  $g \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$ . Substituting  $P = \begin{pmatrix} x & r & \cdots & r \\ 1 & 0 & \cdots & 0 \end{pmatrix} \in \Gamma_n^c[2]$ ,  $Q = (q_1, \dots, q_m) \in \Gamma_m^0[2]$  in (3.18) we get that

$$\left( g \begin{pmatrix} x \\ 1 \end{pmatrix} - M \begin{pmatrix} x \\ 1 \end{pmatrix} \right) \sum_{j=1}^m M(q_j) = 0.$$

Since there exists  $Q^0 \in \Gamma_m^0[2]$  such that  $\sum_{j=1}^m M(q_j^0) \neq 0$  therefore

$$g \begin{pmatrix} x \\ 1 \end{pmatrix} = M \begin{pmatrix} x \\ 1 \end{pmatrix}, \quad x \in ]0, 1[.$$

In a similar way we have that

$$g \begin{pmatrix} 1 \\ y \end{pmatrix} = M \begin{pmatrix} 1 \\ y \end{pmatrix}, \quad y \in ]0, 1[.$$

If there exists  $x_0 \in ]0, 1[$  such that  $g \begin{pmatrix} x_0 \\ 0 \end{pmatrix} \neq 0$  and  $g \begin{pmatrix} 0 \\ y \end{pmatrix} = 0$ ,  $y \in ]0, 1[$  then, by Lemma 2.2, there exists a multiplicative function  $\mu : ]0, 1[ \rightarrow \mathbb{R}$  such that  $M \begin{pmatrix} x \\ y \end{pmatrix} = \mu(x)$ ,  $\begin{pmatrix} x \\ y \end{pmatrix} \in ]0, 1[^2$ . Substituting  $P = \begin{pmatrix} x & r & \cdots & r \\ 1 & 0 & \cdots & 0 \end{pmatrix} \in \Gamma_n^c[2]$ ,  $x \in ]0, 1[$  and  $Q = (q_1, \dots, q_m) \in \Gamma_m^0[2]$  in (3.18), we get that

$$\left( g \begin{pmatrix} x \\ 1 \end{pmatrix} - \mu(x) \right) \sum_{j=1}^m \mu(q_{j1}) = 0.$$

Since there exists  $(q_{11}^0, \dots, q_{m1}^0) \in \Gamma_m^0[1]$  such that  $\sum_{j=1}^m \mu(q_{j1}^0) \neq 0$  thus  $g \begin{pmatrix} x \\ 1 \end{pmatrix} = \mu(x)$ ,  $x \in ]0, 1[$ .

The substitutions  $P = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & r & \cdots & r \end{pmatrix} \in \Gamma_n^c[2]$ ,  $Q = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & s & \cdots & s \end{pmatrix} \in \Gamma_m^c[2]$  in (3.18) imply that  $g \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left( g \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)^2$ , that is,  $g \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \{0, 1\}$ .

If  $g \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$  then, with the substitutions  $P = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ x & r & \cdots & r \end{pmatrix} \in \Gamma_n^c[2]$ ,  $x \in ]0, 1[$ ,  $Q = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & s & \cdots & s \end{pmatrix} \in \Gamma_m^c[2]$  in (3.18), we get that  $g \begin{pmatrix} 1 \\ x \end{pmatrix} = 1$ ,  $x \in ]0, 1[$ .



With the substitutions  $P = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & r & \cdots & r \end{pmatrix} \in \Gamma_n^c[2]$ ,  $Q = \begin{pmatrix} 0 & s & \cdots & s \\ 1 & 0 & \cdots & 0 \end{pmatrix} \in \Gamma_m^c[2]$  in (3.18) we get that  $g\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$ .

With the substitutions  $P = \begin{pmatrix} x & r & \cdots & r \\ 1 & 0 & \cdots & 0 \end{pmatrix} \in \Gamma_n^c[2]$ ,  $x \in ]0, 1[$ ,

$Q = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{pmatrix} \in \Gamma_m^c[2]$  in (3.18) we get that

$$g\begin{pmatrix} x \\ 0 \end{pmatrix} = g\begin{pmatrix} x \\ 1 \end{pmatrix} = \mu(x), \quad x \in ]0, 1[.$$

If  $g\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$  then, with the substitutions  $P = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & r & \cdots & r \end{pmatrix} \in \Gamma_n^c[2]$ ,  $Q = (q_1, \dots, q_m) \in \Gamma_m^c[2]$  in (3.18), we get that  $\sum_{j=1}^m g\begin{pmatrix} q_{j1} \\ 0 \end{pmatrix} = 0$ ,  $(q_{11}, \dots, q_{1m}) \in \Gamma_m^c[1]$ . Thus there exists an additive function  $a : \mathbb{R} \rightarrow \mathbb{R}$  such that  $g\begin{pmatrix} x \\ 0 \end{pmatrix} = a(x) - \frac{a(1)}{m}$ ,  $x \in [0, 1]$ . Since  $0 = g(\underline{0}) = -\frac{a(1)}{m}$  we have that  $a(1) = 0$  and

$$g\begin{pmatrix} x \\ 0 \end{pmatrix} = a(x), \quad x \in [0, 1].$$

With the substitutions  $P = \begin{pmatrix} x & 1-x & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{pmatrix} \in \Gamma_n^c[2]$ ,  $x \in ]0, 1[$ ,  $Q = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{pmatrix} \in \Gamma_m^c[2]$  in (3.18) we get that

$$g\begin{pmatrix} 0 \\ 1 \end{pmatrix} (a(x) + \mu(1-x) - 1) = 0.$$

Since the function  $a$  is additive, the function  $\mu$  is multiplicative and different from the functions  $x \rightarrow 0$ ,  $x \rightarrow 1$ , and  $x \rightarrow x$ , there exists  $x_0 \in ]0, 1[$  such that  $a(x_0) + \mu(1-x_0) \neq 0$  thus

$$g\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0.$$

With the substitutions  $P = \begin{pmatrix} x & 1-x & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{pmatrix} \in \Gamma_n^c[2]$ ,  $x \in ]0, 1[$ ,  $Q = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ y & 1-y & 0 & \cdots & 0 \end{pmatrix} \in \Gamma_m^c[2]$ ,  $y \in ]0, 1[$  in (3.18) we get that  $a(x) = 0$ ,  $x \in ]0, 1[$ . Substituting  $P = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ x & r & \cdots & r \end{pmatrix} \in \Gamma_n^c[2]$ ,  $x \in ]0, 1[$ ,

$Q = \begin{pmatrix} y_1 & y_2 & \cdots & y_m \\ 1 & 0 & \cdots & 0 \end{pmatrix} \in \Gamma_m^c[2]$ ,  $y_1, \dots, y_m \in ]0, 1[$ , in (3.18) we get that

$$\left(g\begin{pmatrix} 1 \\ x \end{pmatrix} - 1\right) \sum_{j=1}^m \mu(y_j) = 0.$$

Since there exists  $(y_1^0, \dots, y_m^0) \in \Gamma_m^0[1]$  such that  $\sum_{j=1}^m \mu(y_j^0) \neq 0$  therefore  
 $g\left(\begin{smallmatrix} 1 \\ x \end{smallmatrix}\right) = 1, x \in ]0, 1[.$  □

## References

- [1] ACZÉL, J. Lectures on Functional Equations and Their Applications, *Academic Press*, New York – London, 1966.
- [2] EBANKS, B. R., SAHOO, P. K., SANDER, W., Characterizations of information measures, *World Scientific*, Singapore – New Jersey – London – Hong Kong, 1998.
- [3] LOSONCZI, L., MAKSA, GY., On some functional equation of information theory, *Acta Math. Acad. Sci. Hungar.*, **39** (1982), 73-82.

**Imre Kocsis**

H-4028, Debrecen

Ótemető u. 2-4.

Hungary